National College of Ireland

School of Computing
First Year Entrance Examination

Mathematics Qualifying Sample Examination
August 2019

SOLUTIONS

Attempt ALL QUESTIONS
All questions carry equal marks

Duration of Exam: 3 hours.
Total Marks: 150
Minimum Pass Marks: 75
Attachments: NCI Undergraduate Formulae Booklet.
Note: Calculators may be used.

Section: Probability and Statistics

Question 1:

(a) A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that the ball drawn is

(i). Red
(ii). White
(iii). Blue
(iv). Not red
(v). Red or white

(15 marks)

Solution (a):

Number of Red Balls = 6
Number of White Balls = 4
Number of Blue Balls = 5
Total Balls = 6 + 4 + 5 = 15

(i). Probability of Red Balls = 6/15
(ii). Probability of White Balls = 4/15
(iii). Probability of Blue Balls = 5/15
(iv). Probability of Not Red Balls = 1 - 6/15 = 9/15
(v). Probability of Red or White Balls = 6/15 + 4/15 = 10/15 = 2/3

(b) Find the probability of 4 turning up at least once in two tosses of a fair die?

(10 marks)

Solution (b):

Sample space: 6 × 6 = 36

$$
\begin{aligned}
S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\
&(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\
&(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\
&(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\
&(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\
&(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}
\end{aligned}
$$

Event (4 turning up at least once): {(1,4),(2,4),(3,4),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,4),(6,4)}

Number of occurrences: 11

Probability of 4 turning up at least once in two tosses: 11/36

Question 2:

Over the course of a 20 day period, a student keeps a record of the number of phone calls that she receives per day. The results are presented in the following frequency distribution:

Number of Phone Calls Received	Number of Days
1	1
2	5
3	3
4	7
5	4

Calculate the **standard deviation** of the distribution.

(25 marks)

Solution 2:

Number of Phone calls Received (x)	Number of Days (fx)	x * f(x)	u	x - u	$(x - u)^2$	$fx * (x - u)^2$
1	1	1	3.4	-2.4	5.76	5.76
2	5	10	3.4	-1.4	1.96	9.8
3	3	9	3.4	-0.4	0.16	0.48
4	7	28	3.4	0.6	0.36	2.52
5	4	20	3.4	1.6	2.56	10.24
	20	**68**				**28.8**
VARIANCE = 28.8/20 = 1.44						
Standard Deviation = 1.2						

u = mean = 68/20 = 3.4
Students can use calculator for the calculations.

Section 2: Geometry and Trigonometry

Question 3:

(a) Find all the values of x for which $\cos(4x) = \sqrt{3}/2$, where $0° < x \leq 360°$.

(10 marks)

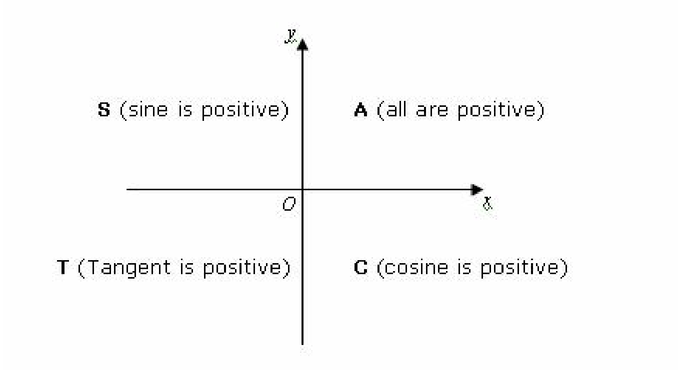
Solution (a):

$$\cos(4x) = \sqrt{3}/2 => 4x = cos^{-1}(\sqrt{3}/2)$$

$$x = \frac{cos^{-1}\left(\sqrt{3}/2\right)}{4} = \frac{15°}{4} = 3.75°$$

In the 4th Quadrant, the 2nd value is

$$x = 360° - 3.75° = 356.25°$$



(b) Find the equation of the perpendicular bisector of the line segment [AB], where A is the point (−14, 10) and B is the point (26, −22).

(15 marks)

Solution (b):

(x1, y1) = (-14, 10)
(x2, y2) = (26, -22)

$$\text{Mid point(M)} = \left(\frac{-14+26}{2}, \frac{10-22}{2}\right)$$
$$\text{Mid point(M)} = (6, -6)$$

$$Slope(m) = \frac{-22-10}{26-(-14)} = \frac{32}{40} = \frac{4}{5}$$

$$y = mx + b => y = \left(\frac{4}{5}\right)x + b$$

The coordinates of the Mid point (6, -6) are used to calculate the equation of perpendicular bisector

$$-6 = \left(\frac{4}{5}\right)6 + b => b = -6 + \frac{24}{5} = \frac{-30+24}{5} = -\frac{6}{5}$$

Substitute the value of b in the slope intercept form of equation ($y = \left(\frac{4}{5}\right)x + b$) as mentioned below

$$y = \left(\frac{4}{5}\right)x - \frac{6}{5} = (4x-6)/5$$
$$5y = 4x - 6$$
$$4x - 5y = 6$$

Section 4: Number Systems

Question 4:

(a) $(4+3i)$ is one root of the equation $az^2 + bz + c = 0$ where $a, b, c \in \mathbb{R}$, and $i^2 = -1$. Write the other root.

(10 marks)

Solution (a):

$$Let\ z = 4 + 3i$$
$$\text{Complex conjugate of } z = 4 - 3i$$
The two roots are (4 + 3i) and (4 – 3i)
$$z^2 + (\text{Sum of roots})z + \text{Product of roots} = 0$$
$$z^2 + (4 + 3i + 4 - 3i)z + (4 + 3i)(4 - 3i) = 0$$
$$z^2 + 8z + (16 - 12i + 12i + 9i^2) = 0$$
$$z^2 + 8z + (16 - 9) = 0 \text{ Where } i^2 = -1$$
$$z^2 + 8z + 7 = 0$$
$$a = 1, b = 8 \text{ and } c = 7 \text{ are all real numbers.}$$
(4 + 3i) and (4 – 3i) are the roots of $z^2 + 8z + 7 = 0$.

(b) Express $z = (3 + 2i)(2 + 2i)$ in polar form and calculate z^4. Express the results both in polar and rectangular forms.

(15 marks)

Solution (b):

$$z = (3 + 2i)(2 + 2i)$$
$$z = 6 + 6i + 4i - 4 = 2 + 10i \text{ (rectangular form)}$$
$$x = 2 \text{ and } y = 10$$
$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (10)^2} = \sqrt{104}$$
$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{10}{2}\right) = 78.6° \cong 79°$$
$$z = \sqrt{104}(\cos(79°) + i\sin(79°)) \text{ (ploar form)}$$

Or

$$z^4 = 10816\big(\cos(79°) + i\sin(79°)\big)^4$$
$$z^4 = 10816(\cos(316°) + i\sin(316°))$$

Section 5: Algebra

Question 5:

(a) Solve the simultaneous equations:

$$x + y + z = 16$$
$$\frac{5}{2}x + y + 10z = 40$$
$$2x + \frac{1}{2}y + 4z = 21$$

to find the values of x, y and z.

(15 marks)

Solution (a):

$x + y + z = 16$ (A)

$5x + 2y + 20z = 80$ (B)

$4x + y + 8z = 42$ (C)

Subtracting 5A - B Equations

$5x + 5y + 5z = 80$ (5A) Multiply A by 5

$5x + 2y + 20z = 80$ (B)

============================

$3y - 15z = 0$ (D) 5A - B

Subtracting 4A - C Equations
$4x + 4y + 4z = 64$ (4A) Multiply A by 4
$4x + y + 8z = 42$ (B)
============================
$3y - 4z = 22$ (E) 4A - B

Subtracting D – E Equations
$3y - 15z = 0$ (D)
$3y - 4z = 22$ (E)
============================
$-11z = -22 => z = 2$ D - E
Substitute value of z in Equation (E)
$3y - 8 = 22 => 3y = 30 => y = 10$
Substitute values of y = 10 and z = 2 in Equation (A)
$x + y + z = 16 => x + 10 + 2 = 16 => x = 4$

Therefore, x = 4, y = 10 and z = 2.

(b) Given the equation $x^2 + (k-2)x + (k-3) = 0$

(i). Show that the roots are real for all values of $k \in \mathbb{R}$.
(ii). Find the roots of the equation in terms of k.

(10 marks)

Solution (b):

(i) $x^2 + (k-2)x + (k-3) = 0$
$b^2 - 4ac = (k-2)^2 - 4 * 1 * (k-3) = k^2 - 4k + 4 - 4k + 12$
$b^2 - 4ac = k^2 - 8k + 16 = (k-4)^2$
For equal roots, $b^2 - 4ac = 0$
For real roots, $(k-4)^2 \geq 0$ Which is TRUE

(ii) So k = 4 is the root of the equation in terms of k.

Section: Functions

Question 6:

(a) A is the closed interval $[0,5]$. That is, $A = \{x \mid 0 \leq x \leq 5, x \in \mathbb{R}\}$. The function f is defined on by

$f: A \to \mathbb{R}$ with $x \mapsto x^3 - 5x^2 + 3x + 5$.

(i). Find the maximum and minimum values of x.
(ii). State whether f is *injective*. Give a reason for your answer.

(15 marks)

Solution (a):

(i).

$$f(x) = x^3 - 5x^2 + 3x + 5$$

$$\text{At } x = 5, f(5) = 5^3 - 5 * 5^2 + 3 * 5 + 5 = 125 - 125 + 15 + 5$$
$$f(0) = 20 \text{ at maximum value of x} = 5$$

$$\text{At } x = 0, f(0) = 0^3 - 5 * 0^2 + 3 * 0 + 5 = 5$$
$$f(0) = 5 \text{ at minimum value of x} = 0$$

(ii).

$$y = f(0) = 0^3 - 5 * 0^2 + 3 * 0 + 5 => \ y = f(0) = 5 \ at \ x = 0$$
$$y = f(1) = 1^3 - 5 * 1^2 + 3 * 1 + 5 => \ y = f(1) = 4 \ at \ x = 1$$
$$y = f(2) = 2^3 - 5 * 2^2 + 3 * 2 + 5 => \ y = f(2) = -1 \ at \ x = 2$$
$$y = f(3) = 3^3 - 5 * 3^2 + 3 * 3 + 5 => \ y = f(3) = -4 \ at \ x = 3$$
$$y = f(4) = 4^3 - 5 * 4^2 + 3 * 4 + 5 => \ y = f(4) = -1 \ at \ x = 4$$
$$y = f(5) = 5^3 - 5 * 5^2 + 3 * 5 + 5 => \ y = f(5) = 20 \ at \ x = 5$$

The ordered pairs (x, y) are obtained based on x values as
(0, 5), (1, 4), (2, -1), (3, -4), (4, -1), (5, 20)

This is not an injective function as more than one element of x maps to the same element (-1).

(b) The equation of a circle is $x^2 + y^2 = 20$. Find $\frac{dy}{dx}$ and hence find the slope of the tangent to the circle at the point $(2,4)$.

(10 marks)

Solution (b):

$$x^2 + y^2 = 20 => \ y^2 = 20 - x^2 => y = \sqrt{20 - x^2}$$
$$2y\frac{dy}{dx} = 0 - 2x => y\frac{dy}{dx} = -x => \frac{dy}{dx} = -\frac{x}{y}$$
$$=> \frac{dy}{dx} = -\frac{x}{\sqrt{20 - x^2}}$$

Slope at x = 2 is

$$\frac{dy}{dx} = -\frac{2}{\sqrt{20 - 4}} = -\frac{2}{4} = -\frac{1}{2}$$